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Comparative study of the critical behavior in one-dimensional random and aperiodic environments*

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Abstract. We consider cooperative processes (quantum spin chains and random walks) in one-dimensional fluctuating random and aperiodic environments characterized by fluctuating exponents $\omega > 0$. At the critical point the random and aperiodic systems scale essentially anisotropically in a similar fashion: length (L) and time (t) scales are related as $L \sim (\ln t)^{1/\omega}$. Also some critical exponents, characterizing the singularities of average quantities, are found to be universal functions of ω, whereas some others do depend on details of the distribution of the disorder. In the off-critical region there is an important difference between the two types of environments: in aperiodic systems there are no extra (Griffiths)-singularities.

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1 Introduction

Quenched disorder often has a strong effect on the cooperative properties of stochastic processes and strongly correlated systems, especially in one space dimension. For example the diffusion process in a one-dimensional environment (Sinai's walk) [1] becomes extremely slow, the mean-square displacement behaves like $[X^2(t)]_{av} \sim \ln^4 t$, in contrast to the linear t-dependence in the homogeneous case.

Ultraslow dynamics has recently also been observed in one-dimensional random quantum spin system [2,3]. The origin of the slow relaxation in these strongly correlated systems is again the presence of quenched disorder, and in particular the vicinity of a quantum critical point. Generally the presence of quenched disorder has a more pronounced effect on quantum phase transitions, which occur at zero temperature and are driven by quantum fluctuations, than on the so-called classical phase transitions, which are driven by thermal fluctuations.

Up to now, these two observations, namely the ultraslow diffusion in general one-dimensional disordered environments and the so-called "quantum" activated dynamics in random quantum spin chains, seemed to be unrelated. One aim of the present paper is to demonstrate a intimate connection between both. Another issue is the question in how far these effects are also present in *aperiodic*, *i.e.* non-random but nevertheless inhomogeneous environments. As we will see, relevant aperiodic systems bear a lot of similarities with completely random systems. However, also crucial differences exist, in particular in the so-called off-critical regime, *i.e.* the Griffiths-McCoy [4,5] region for the quantum spin chains and the anomalous diffusion regime of the random walk. It is simply not existent in aperiodic systems. A short account of the latter issue (random *versus* relevant aperiodicity) has been given elsewhere [6].

Although the results we report are of rather general validity we have, for concreteness and for the lack of space, to confine ourselves to particular examples. So the prototype of a random quantum spin system is the random transverse Ising model (random TIM), which has received considerable interest recently [2,3,7–14]. In one dimension the model is defined by the Hamiltonian

$$H = -\sum_l J_l \sigma_l^x \sigma_{l+1}^x - \sum_l h_l \sigma_l^z \qquad (1.1)$$

in terms of the σ_l^x, σ_l^z Pauli matrices at site l. Here the exchange couplings J_l and the transverse fields h_l

* Dedicated to J. Zittartz on the occasion of his 60th birthday

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are quenched random variables, taken independently from distributions $\pi(J)$ and $\rho(h)$, respectively. The quantum model in (1.1) is closely related to a two-dimensional classical Ising model with randomly layered couplings, which was first introduced and partially solved by McCoy and Wu [15,16] (see also [17–19]). The quantum control parameter of the model is defined by

$$\delta = \frac{[\ln h]_{av} - [\ln J]_{av}}{\mathrm{var[ln\,h]} + \mathrm{var[ln\,J]}}\,. \tag{1.2}$$

For $\delta < 0$ ($\delta > 0$) the system is in the ordered (disordered) phase and at $\delta = 0$ there is a phase transition in the system. One surprising observation is that at the critical point some physical quantities are not *self-averaging*, thus the *typical* and the *average* values of those are different. For these observables the critical properties are determined by the so-called *rare events*, (which occur with vanishing probability) dominating the mean values.

The average magnetization at the surface, m_s and in the bulk, m_b, of the system vanishes as a power law close to the critical point as $m_s \sim \delta^{\beta_s}$ and $m_b \sim \delta^{\beta}$, respectively, with exponents

$$\beta_s = 1 \quad \text{and} \quad \beta = \frac{3-\sqrt{5}}{2}\,. \tag{1.3}$$

The average spin-spin autocorrelation function $G(l) = [\langle\sigma_i^x \sigma_{i+r}^x\rangle]_{av}$ involves the average correlation length ξ, which diverges at the critical point as $\xi \propto |\delta|^{-\nu}$ with the correlation length exponent

$$\nu = 2\,, \tag{1.4}$$

which differs from the exponent of the *typical* correlation function $G_{typ}(l) = \exp([\ln\langle\sigma_i^x \sigma_{i+r}^x\rangle]_{av}\}$, which is

$$\nu_{typ} = 1. \tag{1.5}$$

The time-dependent correlations of the model are very special, they differ completely from those in the pure system. The autocorrelation function

$$G_l(t) = [\langle\sigma_l^x(0)\sigma_l^x(t)\rangle]_{av} \tag{1.6}$$

at the critical point decays on a logarithmic scale

$$G_l(t) \sim (\ln t)^{-\eta} \quad (\delta = 0), \tag{1.7}$$

where the decay exponent η satisfies the scaling relation $\eta = \beta/\nu$ and $\eta_s = \beta_s/\nu$ for site l in the bulk and on the surface, respectively. Leaving the critical point in any direction one enters the Griffiths-McCoy regions, in which the connected autocorrelation function has a power law decay:

$$G_l(t) \sim t^{-z(\delta)} \quad (\delta \neq 0), \tag{1.8}$$

where the dynamical exponent $z(\delta)$ is a continuous function of the parameter δ. Close to the critical point it is $z(\delta) \approx 1/2\delta$ [7]. As a consequence of the power-law decay of the autocorrelations in the Griffiths-McCoy phase the magnetization is a singular function of the uniform magnetic field H_x as $m_{sing} \propto |H_x|^{1/z(\delta)}$.

The above results about the random TIM are independent of the actual form of the probability distribution. It is often argued that the perturbation caused by the disorder, with respect to the pure system, is connected to the fluctuating energy per spin:

$$\Delta(L)/L = \frac{1}{L}\sum_{l=1}^{L}(J_l - [J]_{av}) \sim L^{\omega-1}. \tag{1.9}$$

Here L is the linear size of the sequence of couplings J_l under consideration and $\omega = \omega_{rand} = 1/2$ is the fluctuating or wandering exponent. The Harris criterion [20] makes a statement on behalf of the relevance/irrelevance of the perturbation of the critical behavior of the pure system by the above disorder. One compares the strength of the thermal fluctuations at the critical point with the fluctuating energy in (1.9) on a length scale L identical to the correlation length. This yields in a one-dimensional system $\Phi = 1 + \nu(\omega - 1)$ for the cross-over exponent [21,22] and indeed, for the random sequence the perturbation is relevant, since with $\nu = 1$ and $\omega = 1/2$ one gets $\Phi = 1/2 > 0$.

It is known that there are non-random deterministic sequences, generated through substitutional rules, which have unbounded fluctuations, so that the corresponding fluctuating exponent is $\omega > 0$. Having this similarity between random and aperiodic sequences in mind, one might ask the question whether or not the fluctuating exponent ω is the only quantity that determines the critical behavior of systems with unbounded fluctuations in the couplings.

As an example we consider the Rudin–Shapiro (RS) sequence [23], which is built on four letters A, B, C and D with the substitutional rule:

$$\mathrm{A} \to \mathrm{AB}, \quad \mathrm{B} \to \mathrm{AC}, \quad \mathrm{C} \to \mathrm{DB}, \quad \mathrm{D} \to \mathrm{DC}. \tag{1.10}$$

Thus starting with a letter A one proceeds as: A $\to$ AB $\to$ ABAC $\to$ ABACABDB $\to$ *etc.*, and one may assign different couplings to the different letters. The fluctuating exponent of the RS-sequence, which was originally introduced to mimic random fluctuations, $\omega_{RS} = 1/2$, *i.e.* just the same as for the random sequence.

We note that up to now the critical behavior of aperiodic systems has been studied mainly for such distributions that are *non-relevant* in the sense of the Harris criterion (*i.e.* $\omega \leq 0$). Especially for marginal sequences (*i.e.* $\omega = 0$) non-universal critical behavior coupling dependent anisotropy exponents have been found in exact calculations for the transverse field Ising chain [24–27].

In the relevant case, $\omega > 0$, there are only a few exact results, obtained for one specific representation described after equation (1.10) starting with the letter A. For this case the magnetization of finite RS-chains of length L at the critical point has been shown to behave at the two end points as [28]

$$m_s(\mathrm{RS}) \sim \exp(-const.\ \sqrt{L}) \tag{1.11}$$

and $\overline{m_s}(\mathrm{RS}) = const. > 0$. Moreover, the lowest excitation energy at the critical point is given by

$$\varepsilon_1(L) \sim \exp(-const.\ \sqrt{L}). \tag{1.12}$$

We also note on recent Monte-Carlo simulations on the critical behavior of the $Q = 8$-state Potts model with layered aperiodic modulations [29].

In the present paper we are going to study the *average quantities* also for aperiodic chains. This means that for instance for the bulk magnetization we average over all sites of the system. This averaging process, however, could be performed in another way. One generates (hypothetically) an infinite sequence through substitution, cut out segments of length L starting at all different positions of the (infinite) sequence and then average over all segments. Then, considering the example of surface magnetization one could expect that the two generic situations in and below equation (1.11) will appear with given probabilities, which are then connected to the average critical behavior of the system.

The structure of the paper is the following: in Section 2 we present the basic notations of aperiodic sequences used in the paper. The free fermionic description of the TIM is given in Section 3. Results about the TIM both at the critical point and in the off-critical region are presented in Sections 4 and 5, respectively. We conclude with a final section on a parallel analysis of the behavior of a random walk in random or aperiodic environments.

2 Aperiodic substitutional sequences

In this paper we consider sequences generated *via* substitution on a finite alphabet such that, in the case of two letters A and B, one substitutes $\mathrm{A} \to S(\mathrm{A})$ and $\mathrm{B} \to S(\mathrm{B})$. For the RS-sequence the explicit relations are given in equation (1.10). The properties of the sequence are governed by the substitutional matrix:

$$\mathbf{M} = \begin{pmatrix} n_{\mathrm{A}}^{S(\mathrm{A})} & n_{\mathrm{A}}^{S(\mathrm{B})} \\ n_{\mathrm{B}}^{S(\mathrm{A})} & n_{\mathrm{B}}^{S(\mathrm{B})} \end{pmatrix} \tag{2.1}$$

where the matrix element $n_i^{S(j)}$ is the number of the letters i occurring in the substitution $S(j)$. The matrix elements of $\mathbf{M}^n$ contain the same numbers in the sequence after n iterations.

If $\mathbf{U}_r$ denotes the right eigenvector of $\mathbf{M}$ with eigenvalues Ω_r, the asymptotic density of i is given by

$$\rho_\infty^{(i)} = \frac{\mathbf{U}_1(i)}{\sum_j \mathbf{U}_1(j)}, \tag{2.2}$$

where $\mathbf{U}_1$ is the eigenvector corresponding to the leading eigenvector Ω_1. The length of the sequence after n iterations is related to the leading eigenvalue through

$$L_n \sim \Omega_1^n. \tag{2.3}$$

We consider the TIM in (1.1) with a constant transverse field $h_l = h$ in an aperiodic environment. We assign to each letter in the sequence a coupling constant (for example J_{A} and J_{B}). The cumulated deviation $\Delta(L)$ from the average coupling $[J]_{av}$ in equation (1.9) scales with L as

$$\Delta(L) \propto |\Omega_2|^n \propto L^\omega\ , \tag{2.4}$$

where Ω_2 is the next-to-leading eigenvalue of the substitution matrix $\mathbf{M}$ and ω is the fluctuating exponent already defined in (1.9). Combining (2.4, 2.3) one gets

$$\omega = \frac{\ln|\Omega_2|}{\ln \Omega_1}\ . \tag{2.5}$$

The critical point h_c of the system is obtained from the relation (1.2) with $\delta = 0$ as

$$\sum_i \rho_\infty^{(i)} \ln J_i = \ln h_c. \tag{2.6}$$

In the following we introduce a family of relevant sequences defined on k-letters (k-general sequence) $\mathrm{A}_1, \mathrm{A}_2, \ldots, \mathrm{A}_k$ with the substitutional rules:

$$\begin{array}{lll} \mathrm{A}_1 \to \mathrm{A}_1\mathrm{A}_2 & & \\ \mathrm{A}_i \to \mathrm{A}_{i-1}\mathrm{A}_{i+1} & & 1 < i \le k/2 \\ \mathrm{A}_i \to \mathrm{A}_{i+1}\mathrm{A}_{i-1} & & k/2 < i < k \\ \mathrm{A}_k \to \mathrm{A}_k\mathrm{A}_{k-1}. & & \end{array} \tag{2.7}$$

The substitutional matrix of the sequence is given by

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & & & & & \\ 1 & 0 & 1 & & & & \\ & 1 & 0 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & 1 & 0 & 1 \\ & & & & & 1 & 1 \end{pmatrix} \tag{2.8}$$

with the eigenvalues

$$\Omega_r = 2\cos[\frac{\pi}{k}(r-1)] \quad r = 1, 2, \ldots, k. \tag{2.9}$$

Thus the fluctuating exponent ω from (2.5) for k-general sequences is given by

$$\omega_k = \frac{\ln[2\cos(\pi/k)]}{\ln 2} \tag{2.10}$$

and is positive for $k \ge 4$. The leading right eigenvector of $\mathbf{M}$ is just $\mathbf{U}_1(i) = const.$, thus the asymptotic density of letters is given by

$$\rho_\infty^{(i)} = 1/k. \tag{2.11}$$

We note that the k-general sequence for $k = 2$ is just the Thue-Morse sequence, while for $k = 4$ we recover the RS-sequence mentioned in the Introduction in (1.10).

To complete this section on the introduction of aperiodic sequences we have to discuss the averaging process. As described in the introduction the average for aperiodic sequences is made in such a way that from a (hypothetical) infinite sequence segments of length L are cut out starting at all different points in the infinite sequence, these we call the realizations of length L of the aperiodic sequence. Then we average physical observables over these realizations. The number of different realizations $R(L)$ of a *random* (binary) sequence grows exponentially with the size of the system, whereas in the *aperiodic* case have a much slower, namely linear increase:

$$R(L) = aL \tag{2.12}$$

with a being some constant. For the RS-chain the number of different realizations is less than $16L$, and one can obtain the exact average value by the following procedure. Consider the RS-sequence in (1.10) generated from a letter A, take the first $4L$ sequels of length L, starting at positions $1, 2, \ldots, 4L$ and take also their reflection symmetric counterparts. Then repeat the procedure starting with a letter D. The averaging then should be performed over these $16L$ realizations. Therefore it is possible in numerical computations to do an *exact* average for the RS-sequence, taking into account *all* different realizations and to obtain numerically exact results for relatively large ($L \leq 512$) aperiodic chains. For other values of k in the k-general case the number of different realizations is still linear in L, however, no similar simple rule to generate all of them in linear computing time is known to us.

3 Free fermionic representation of the TIM

We consider the TIM in equation (1.1) on a finite chain of length L with free or fixed boundary conditions, thus we set $J_L = 0$. The simplest way to calculate various physical quantities of the model is to transform it into a fermionic representation according to Lieb, Schultz and Mattis [30]. A detailed description of the method and its application to different physical quantities can be found in [2], hereafter referred to as paper I. Here we briefly recapitulate the main results.

The Hamiltonian (1.1) is expressed in terms of fermion creation and annihilation operators η_q^+ and η_q, respectively:

$$H = \sum_{q=1}^{L} \varepsilon_q \left(\eta_q^+ \eta_q - \frac{1}{2} \right), \tag{3.1}$$

where the fermionic energies ε_q are identical to the non-negative eigenvalues of the $2L \times 2L$ tridiagonal matrix $\mathbf{T}$ with eigenvectors V_q:

$$\mathbf{T} = \begin{pmatrix} 0 & h_1 & & & & & \\ h_1 & 0 & J_1 & & & & \\ & J_1 & 0 & h_2 & & & \\ & & h_2 & 0 & \ddots & & \\ & & & \ddots & \ddots & J_{L-1} & \\ & & & & J_{L-1} & 0 & h_L \\ & & & & & h_L & 0 \end{pmatrix},$$

$$V_q = \begin{pmatrix} - & \Phi_q(1) \\ & \Psi_q(1) \\ - & \Phi_q(2) \\ & \vdots \\ & \Psi_q(L-1) \\ - & \Phi_q(L) \\ & \Psi_q(L) \end{pmatrix}. \tag{3.2}$$

The magnetization profiles

$$m_l = \langle 0 | \sigma_l^x | 0 \rangle \tag{3.3}$$

for symmetry braking boundary conditions (b.c.) can be expressed as $l \times l$ determinants, whose elements are defined in terms of the components $\Psi_q(i)$ and $\Phi_q(i)$ of the eigenvectors V_q in (3.2). If we fix the spin at one end of the chain, which is equivalent to setting $h_L = 0$, the local magnetization is given by

$$m_l^{free} = \begin{vmatrix} H_1 & G_{11} & G_{12} & \ldots & G_{1l-1} \\ H_2 & G_{21} & G_{22} & \ldots & G_{2l-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_l & G_{l1} & G_{l2} & \ldots & G_{ll-1} \end{vmatrix}, \tag{3.4}$$

where

$$\begin{aligned} H_j &= \Phi_1(j) \\ G_{jk} &= -\sum_q \Psi_q(k) \Phi_q(j). \end{aligned} \tag{3.5}$$

The surface magnetization for fixed-spin b.c. $m_s = m_1$ is given by the very simple formula

$$m_s = \left[1 + \sum_{l=1}^{L-1} \prod_{j=1}^{l} \left(\frac{h_j}{J_j} \right)^2 \right]^{-1/2}, \tag{3.6}$$

which is *exact* for any *finite* system.

The magnetization profile can be calculated analogously (see paper I), when both surface spins are fixed ($++$ and $+-$ b.c.s). Finally , for a finite open chain the ground state expectation value of σ_l^x in (3.3) is identically zero, due to symmetry. Here, as shown in [10] the matrix-element profile

$$m_l^{free} = \langle 1 | \sigma_l^x | 0 \rangle \tag{3.7}$$

is of importance, which is formally given by equation (3.4) with $h_L \neq 0$.

Here we note on a simple estimate for the excitation energy $\varepsilon_1(L)$ in a open chain of length L [26]:

$$\varepsilon_1(L) \sim m_s \overline{m}_s \; h_L \prod_{i=1}^{L-1} \frac{h_i}{J_i} \cdot \tag{3.8}$$

provided $\lim_{L\to\infty} \varepsilon_1(L) L = 0$. Here m_s and $\overline{m}_s$ denote the finite size surface magnetizations at both ends of the chain, as defined in equation (3.6) (for $\overline{m}_s$ simply replacing h_j/J_j by h_{L-j}/J_{L-j} in this equation).

Next we consider the dynamical correlation functions of the system as a function of the imaginary time τ:

$$G_l(\tau) = \langle 0 | \sigma_l^x(\tau) \sigma_l^x(0) | 0 \rangle \; . \tag{3.9}$$

For surface spins this can be expressed in the simple form

$$G_1(\tau) = \sum_q |\Phi_q(1)|^2 \exp(-\tau \varepsilon_q) \; . \tag{3.10}$$

whereas in the bulk $G_l(\tau)$ can be expressed as a Pfaffian that can be evaluated *via* a determinant of $l \times l$ antisymmetric matrix (see paper I for details).

4 Critical properties

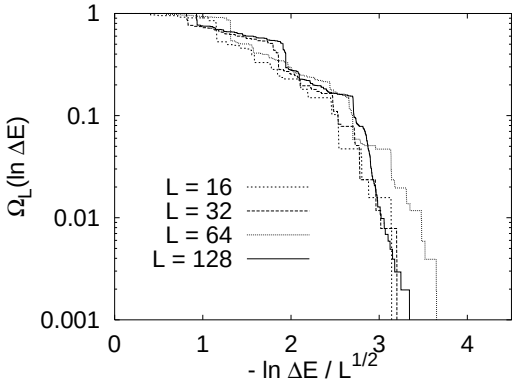
In what follows we take for the Hamiltonian (1.1) homogeneous fields $h_l = h$ and two-valued couplings, say $J_l = \lambda$ and $J_l = 1/\lambda$. For the k-general sequence in (2.7) $J_l = \lambda$ for letters A_i with $i < (k+1)/2$ and $J_l = 1/\lambda$ for letters with $i > (k+1)/2$. For an odd k we take $J_l = 1$ for $i = (k+1)/2$. Thus for the RS-sequence in (1.10) we have $J_\mathrm{A} = J_\mathrm{B} = \lambda$ and $J_C = J_D = 1/\lambda$. Then from (2.6) and (2.11) for the critical point follows:

$$\delta = \ln h_c = 0, \tag{4.1}$$

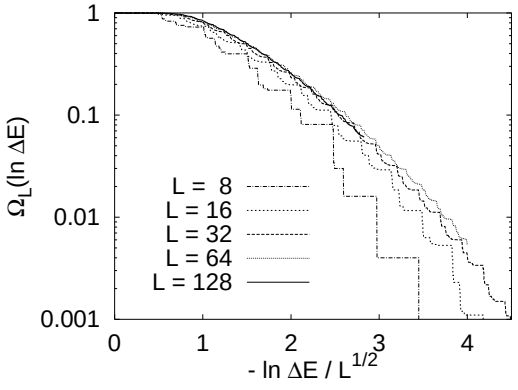
independently of k. Similarly, for the random model we take homogeneous fields and binary distribution of the couplings: $J_l = \lambda$ and $J_l = 1/\lambda$, with the same probability. The critical point in this case is also given by equation (4.1).

4.1 The distribution of low energy excitations

The basic features of the random transverse Ising chain at criticality have been described in the introduction. Here we stress again that many unusual properties of the random TIM are connected to the extremely broad distribution of the different physical quantities at the critical point. Concerning the energy gap ΔE in a finite system of size L the appropriate scaling variable is $\ln \Delta E / \sqrt{L}$. The reason for this is that the system is essentially anisotropic at criticality, which means that times and length scale are related in an exponential rather than algebraic manner, *cf.* equation (1.12).



(a)



(b)

Fig. 1. (a) Scaling plot of the integrated probability density $\Omega(\ln \Delta E)$ *versus* the scaling variable $(\ln \Delta E / L^{1/2})$ for the RS-sequence (exact average) with $\lambda = 4$, *cf.* equation (4.3). (b) The same as in (a) for the the random sequence.

For relevant aperiodic systems (*i.e.* those with a positive fluctuating exponent ω) one expects from scaling considerations [21,22,27,28] that the energy gap at the critical point has a similar scaling relation as in random systems:

$$\Delta E(L) \sim \exp(-const. \; L^\omega). \tag{4.2}$$

This form actually follows from the formula in equation (3.8) for the energy gap of a sample with local order at the two ends, *i.e.* $m_s = \mathcal{O}(1)$ and $\overline{m}_s = \mathcal{O}(1)$, where m_s and $\overline{m}_s$ are the surface magnetizations at the left and right end, respectively, to be calculated with equation (3.6). Then we have $\varepsilon_1 \sim \prod_{i=1}^{l-1} \frac{h_i}{J_i} \sim \exp(-l_{tr} \overline{\ln(J/h)})$, where $l_{tr} \sim L^\omega$ measures the size of the transverse fluctuations and $\overline{\ln(J/h)}$ is an averaged reduced coupling, from which (4.2) follows.

To check the validity of the above scaling relation (4.2) we have investigated the probability distribution of the energy gap $P_L(\Delta E)$ at the critical point of the RS-chain and compared it with the same quantity for the random chain.

According to equation (4.2) the appropriate scaling variable is $\ln \Delta E/L^{1/2}$ for both systems. Indeed, as seen in Figure 1a, the integrate probability distribution function

$$\Omega_L(\ln \Delta E) = \int_{-\infty}^{\ln \Delta E} dy\, \tilde{P}_L(y) \sim \tilde{\Omega}(\ln \Delta E/\sqrt{L}) \quad (4.3)$$

with $\tilde{P}_L(\ln \Delta E) = P_L(\Delta E)\Delta E$, has a good data collapse using this reduced variable. Considering the same quantity for the random chain one can observe the same scaling behavior, as shown in Figure 1b. Thus we can conclude, that both the random and the RS-chains have logarithmically broad distribution of the energy gaps at the critical point, from this fact one expects similar consequences for the critical behavior in the two systems.

4.2 Surface magnetization

For the TIM the surface magnetization represents perhaps the most simple order-parameter of the system, which in the fix-free b.c. is explicitly given by the simple formula in (3.6). For one single RS-chain, starting with the letter A this expression has been exactly evaluated at the critical point with the following results [28].

For $\lambda > 1$, when the average couplings at the surface are stronger than in the bulk, the surface is ordered at the critical point, so that the surface magnetization is finite:

$$\lim_{L\to\infty} m_s(L, \lambda, h = 1) = \frac{\lambda^2 - 1}{\sqrt{\lambda^4 - \lambda^2 + 1}} \qquad \lambda > 1 \quad (4.4)$$

and approaches unity in the limit $\lambda \to \infty$. On the other hand for $\lambda < 1$, when the average couplings at the surface are weaker than in the bulk the critical surface magnetization vanishes as

$$m_s(L, \lambda, h = 1) \sim \exp(-const.\ \sqrt{L}). \quad (4.5)$$

Proceeding by studying the critical point magnetizations in other sequels of the RS-chain one can notice that the above two examples are generic: in a sample the surface magnetization is either finite $m_s = \mathcal{O}(1)$, or it vanishes in the stretched exponential form as in equation (4.5). Therefore the average is dominated by the sample with finite surface magnetization ("rare events"), which occur with probability $P_{rare}(L) \propto L^{-\gamma}$. Thus the critical surface magnetization is *not self-averaging*, it is determined by rare events and its scaling dimension x_m^s defined by the asymptotic relation $[m_s(L, h)]_{av} \sim L^{-x_m}$ is just $x_m^s = \gamma$.

In the following we make extended use of this observation and calculated x_m^s exactly. Here we adopt the random walk picture of paper I. *First* we assign to each sample with a given realization of couplings a walk, which starts at the origin and makes the lth step $+1$ (-1) for a coupling $J_l = \lambda$ $(J_l = 1/\lambda)$. *Second*, we take the limit $\lambda \to \infty$, in which only those samples have non-vanishing surface magnetization, where the corresponding walk never visits sites with negative coordinates. Thus the proportion of rare events is given by the survival probability of the walk $P_{rare}(L) = P_{surv}(L)$. Thus the *third* point in the study is to calculate the surviving probability of the walker. For the random chain the corresponding random walk is characterized by a surviving probability of $P_{surv} \sim L^{-1/2}$, thus one gets the exact result:

$$x_m^s(\text{rand}) = 1/2. \quad (4.6)$$

For the RS-chain we have performed the exact analysis of the average surface magnetization, the result of which is presented in the appendix. The leading L-dependence of it is given by

$$[m_s(L, \lambda \to \infty, h = 1)]_{av} = \frac{5}{8}\left(\frac{1}{\sqrt{2}} + \frac{1}{4}\right) L^{-1/2} + \mathcal{O}(L^{-3/4}) \quad (4.7)$$

from which the value of the scaling dimension

$$x_m^s(RS) = 1/2. \quad (4.8)$$

follows. Thus we conclude that the surface magnetization scaling dimension is the same for the random and the RS-chain.

In what follows we show that for relevant aperiodic sequences x_m^s is a simple function of the fluctuating exponent ω. We consider the scaling behavior of the surviving probability $P_{surv}(L)$ of the corresponding aperiodic walk, performing a discrete scale transformation, which corresponds to a substitutional step of the sequence. Then the length of the walk scales as $L \to L\Omega_1$, whereas the transverse fluctuations l_{tr} scale like $l_{tr} \to l_{tr}|\Omega_2|$. Then the number of surviving walks with L steps, $N(L)$, scales as $N(L) \to N(L\Omega_1) = |\Omega_2|N(L)$, since the number of these walks is proportional to the size of the transverse fluctuations. Remembering that the total number of different sequels is $R(L) = aL$ the survival probability $P_{surv}(L) = N(L)/R(L)$ satisfies the scaling relation

$$P_{surv}(\Omega_1 L) = \frac{|\Omega_2|}{\Omega_1} P_{surv}(L) \quad \Rightarrow P_{surv}(L) \propto L^{-(1-\omega)} \quad (4.9)$$

from which the value of the surface magnetization scaling dimension can be read off as

$$x_m^s = 1 - \omega. \quad (4.10)$$

We have seen by an exact analytical treatment that this relation is satisfied for the random and the RS-chain. For the first few members of the family of k-general sequences we verified equation (4.10) numerically and obtained $x_m^s(k = 5) = 0.3059(5)$ and $x_m^s(k = 6) = 0.2076(4)$, which is in good agreement with the corresponding prediction from (4.10) $x_m^s(k = 5) = 0.3058$ and $x_m^s(k = 6) = 0.2075$, respectively.

Now we follow the analysis of paper I and calculate the correlation length critical exponent ν from the $\delta = \ln h$ dependence of the surface magnetization. In the scaling limit $L \gg 1$, $|\delta| \ll 1$ the surface magnetization can be written

as $[m_s(L,\delta)]_{av} = [m_s(L,0)]_{av}\tilde{m}_s(\delta L^{1/\nu})$. Expanding the scaling function into a Taylor series $\tilde{m}_s(z) = 1+bz+O(z^2)$ one obtains for the δ correction to the surface magnetization:

$$[m_s(L,\delta)]_{av} - [m_s(L,0)]_{av} \sim \delta L^{\Theta} \tag{4.11}$$

with

$$\Theta = 1/\nu - x^s_m. \tag{4.12}$$

This exponent can also be determined exactly in the $\lambda \to \infty$ limit from random walk arguments. As shown in paper I the surface magnetization of *rare events* is given by:

$$m_s(L,\delta) = (1+n)^{-1/2} - \delta\frac{\sum_{i=1}^{n} l_i}{(n+1)^{3/2}} + O(\delta^2)\ , \tag{4.13}$$

where the corresponding surviving walk returns n-times to its starting point after $l_1, l_2, \ldots, l_n$ steps. The largest contribution to the coefficient of the term proportional to δ in (4.13) comes from those surviving walks, that have a large n, *i.e.* which visit the starting point frequently. For general aperiodic sequences n grows with the length of the walk as

$$n \sim L^{d_s} \tag{4.14}$$

and we call d_s the surface fractal dimension of the surviving walks. One can check for the k-general sequences that after two substitutional steps, when $L \to 4L$, the number of return points scales as $n \to 2n$, thus from equation (4.14) we get

$$d_s = 1/2 \tag{4.15}$$

independent of the value $k \geq 4$.

Next we are going to perform the average of the linear term in (4.13). Here we note that among the $R(L)$ different samples there are $D(L) = O(n)$ that deliver the same dominant contribution: each of those has $N_R(L) = O(n)$ return points of characteristic length $l_i \sim l_{char}(L) = \mathcal{O}(L)$. Thus the average of the linear term in (4.13) grows like $D(L)\ l_{char}(L)\ N_R(L)/(R(L)\ n^{3/2}) \sim L^{d_s/2}$. Hence, comparing with (4.11) one gets the exponent relation:

$$\frac{1}{\nu} - x^s_m = \frac{d_s}{2}\cdot \tag{4.16}$$

The random case $\nu(\mathrm{rand}) = 2$ is formally contained in (4.16) with $d_s = 0$, since a surviving random walk returns $n = O(1)$-times to the starting point. For the family of k-general sequences $d_s = 1/2$ for all values of k, thus the corresponding correlation length exponent is $\nu(k) = 4/(5-4\omega_k)$, with ω_k given in (2.10). In particular for the RS-sequence we get

$$\nu(\mathrm{RS}) = 4/3. \tag{4.17}$$

We have checked this relation numerically by evaluating exactly the average surface magnetization up to $L = 2^{21}$.

Table 1. Numerical estimates of the exponent Θ in (4.11) comparing numerically exact results on finite systems of sizes 2^{2l-1} and 2^{2l+1}.

l	Θ
2	0.19172235
3	0.15877113
4	0.16048277
5	0.17656409
6	0.19500760
7	0.21068687
8	0.22247716
9	0.23086745
10	0.23670532

The two point fits for the exponent Θ in (4.11) comparing systems of size 2^{2l-1} and 2^{2l+1} are given in Table 1. Using standard sequence extrapolation techniques [31] on the data in Table 1 we got an estimate $\Theta = 0.2501(2)$ which is in excellent agreement with the scaling result (4.12) with (4.17).

For the other members of the k-general sequence the numerical estimates are $\nu(k=5) = 1.82(5)$ and $\nu(k=6) = 2.23(5)$, what should be compared with the scaling predictions $\nu(k=5) = 1.799$ and $\nu(k=6) = 2.186$, obtained from equation (4.16).

4.3 Magnetization profiles

In a geometrically constrained finite system at the critical point the appropriate way to describe the position dependent physical quantities, such as magnetization or energy density, to use density profiles rather then bulk and surface observables [32]. For two-dimensional classical and one-dimensional quantum systems with homogeneous couplings conformal invariance provides a useful tool to study various geometries. Let us consider a critical system confined between two parallel plates, which are at large, but finite distance L apart, where the local densities $\phi(r)$ vary with the distance l from the plates as a smooth function of l/L. According to conformal invariance [33]

$$\langle\phi(l)\rangle_{ab} = \left[\frac{L}{\pi}\sin\pi\frac{l}{L}\right]^{-x_\phi} G_{ab}(l/L), \tag{4.18}$$

where x_ϕ is the scaling dimension of the operator ϕ and the scaling function $G_{ab}(x)$ depends on the universality class of the model and on the type of the boundary conditions to the left a and to the right b. With symmetric b.c. the scaling function is constant $G_{aa} = A$.

In two dimensions conformal invariance can also be used to predict the critical off-diagonal matrix elements profiles $\langle\phi|\phi(l)|0\rangle$, where $\langle\phi|$ denotes the lowest excited state leading to a non-vanishing matrix element (see [2,10]). These off-diagonal profiles give information about the surface and bulk critical behavior *via* finite size

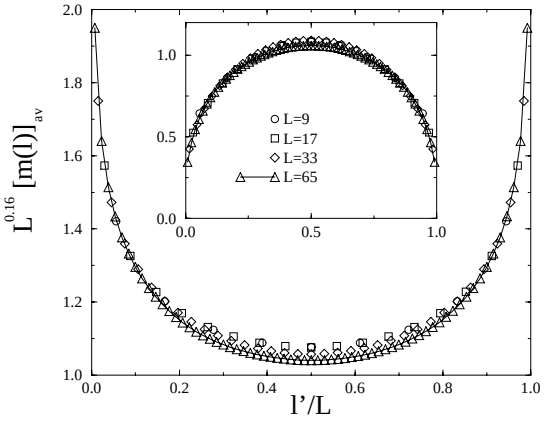


Fig. 2. Scaling plot of the magnetization profile $m_L(l)$ with symmetric (fixed at both ends) boundary condition (4.18) for the RS-sequence. The inset shows a scaling plot of the off-diagonal matrix element (4.19) with symmetric (free at both ends) boundary condition. Here it $\lambda = 4$, the bulk magnetization exponent is used as a fit parameter: $x_m = 0.160(1)$ gives the best data collapse.

scaling. With symmetric b.c. one obtains for the profile [34]

$$\langle\phi|\phi(l)|0\rangle \left(\frac{\pi}{\mathrm{L}}\right)^{x_\phi} \left[\sin\pi\frac{l}{L}\right]^{x_\phi^s - x_\phi} \tag{4.19}$$

which involves also the surface scaling dimension x_ϕ^s.

For the *random* TIM model several profiles have been calculated in [2,10] and they all follow very well the conformal predictions. This coincidence of the numerical and the conformal results is quite surprising, since the random TIM is not conformally invariant due to strongly anisotropic scaling at the critical point.

Here we consider the same problem for aperiodic chains and calculate the magnetization profiles for the RS-chain. In Figure 2 we present the scaling plot of the diagonal and off-diagonal profiles with symmetric boundary conditions. Here we take the exact value for the surface magnetization scaling dimension x_m^s from equation (4.8), whereas the bulk magnetization exponent, x_m, is used as a fit parameter in order to obtain a good data collapse. As seen in Figure 2 both profiles can be fitted with the same exponent

$$x_m(\mathrm{RS}) = 0.160(5), \tag{4.20}$$

which turned out to be independent of the parameter λ and is different from the random chain value predicted by Fisher [7] to be $x_m(\mathrm{rand}) = (3-\sqrt{5})/4 = 0.191\ldots$ The profiles, however, in analogy to the random case, follow very well the conformal predictions, both for the diagonal and off-diagonal profiles (4.18, 4.19). This is an unexpected result, if we take into account that the relevantly aperiodic Ising chains are not conformally invariant, due to anisotropic scaling at the critical point.

4.4 Dynamical correlations

Here we consider (imaginary) time dependent correlations of the same spin, as defined in equation (1.6). One expects different types of asymptotic behavior of the surface spins and of the bulk spins. First we consider the bulk autocorrelation function

$$G(\tau) = [\langle\sigma_{L/2}^x(\tau)\sigma_{L/2}^x\rangle]_{av} \tag{4.21}$$

and present a scaling consideration, where we essentially follow the steps of reasoning in the random case [2,3].

The autocorrelation function, like to the (local) magnetization, is not self-averaging at the critical point: its average value is determined by the *rare events*, which occur with a probability P_r and P_r vanishes in the thermodynamic limit. In the random quantum systems the disorder is strictly correlated along the time axis, consequently in the rare events with a local order, *i.e.* with a finite magnetization also the autocorrelations are non-vanishing. Under a scaling transformation, when lengths are rescaled as $l' = l/b$, with $b > 1$ the probability of the rare events transforms as $P_r' = b^{-x_m}$, like to the local magnetization. As we said above the same is true for the autocorrelation function:

$$G(\ln\tau) = b^{-x_m} G(\ln\tau/b^\omega) \qquad \delta = 0, \tag{4.22}$$

where we have made use of the relation between relevant time t_r and length ξ at the critical point, which follows from the scaling relation in equation (4.2). Setting the rescaling factor to $b = (\ln\tau)^{1/\omega}$ we obtain:

$$G(\tau) \sim (\ln\tau)^{-x_m/\omega}. \tag{4.23}$$

For surface spins in (4.22, 4.23) one should use the surface magnetization scaling dimension, x_m^s.

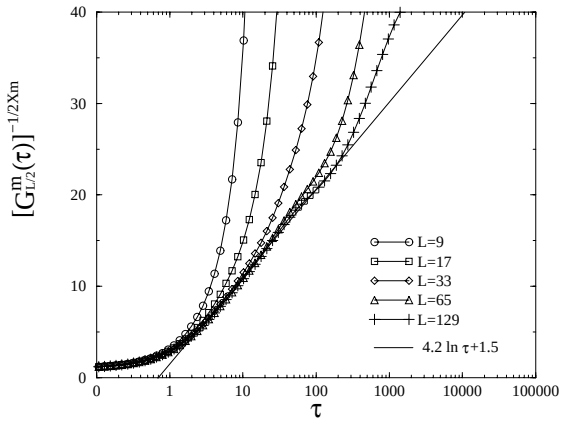
We have numerically calculated the magnetization autocorrelation function both at the surface and in the bulk for the RS-sequence. The results are depicted in Figure 3. As can be seen the finite lattice results collapse onto one single curve (manifesting the absence of finite size effects) and the critical temporal decay happens on a logarithmic scale. The corresponding decay exponents are given by x_m/ω (x_m^s/ω), for bulk (surface) correlations in agreement with the scaling result in (4.23).

Next we consider the scaling behavior of the energy-energy autocorrelation function $G_l^e(\tau) = [\langle\sigma_l^z(\tau)\sigma_l^z\rangle]_{av}$. We note that σ_l^z represents one part of the local energy operator, the other part of which – $\sigma_l^x\sigma_{l+1}^z$ – is related to it through duality, see paper I. Therefore the above autocorrelation function has essentially the same scaling behavior as the full energy density.

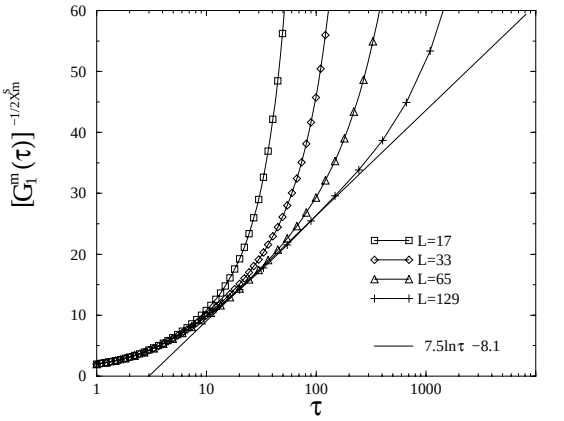
For the random chain, as was shown in I, the critical energy autocorrelation function has an asymptotic power law decay,

$$G_l^e(\tau) \sim \tau^{-\eta_e} \tag{4.24}$$

with the critical exponents $\eta_e(\mathrm{rand}) \approx 2.2$ in the bulk and $\eta_e^s(\mathrm{rand}) \approx 2.5$ on the surface. For the aperiodic RS-chain

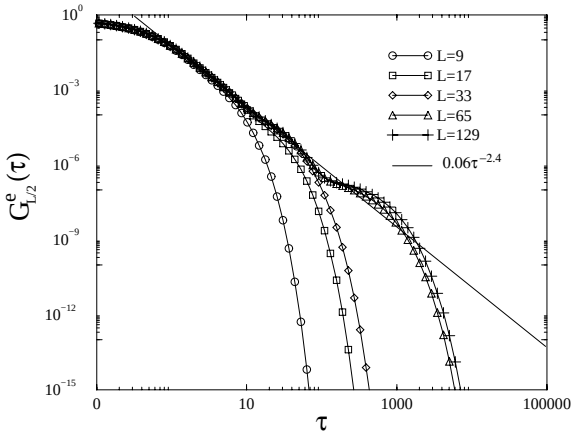


(a)

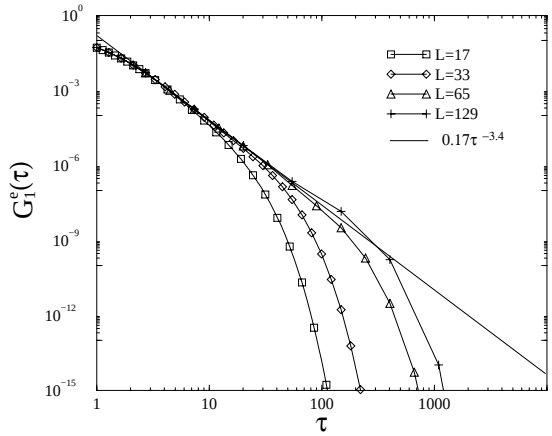


(b)

Fig. 3. (a) Bulk spin-spin autocorrelation function of the RS-sequence $G^m_{L/2}(\tau) = [\langle\sigma^x_{L/2}(t)\sigma^x_{L/2}\rangle]_{av}$ in imaginary time for various system sizes (and $\lambda = 4$). Note that we have chosen L to be odd, so that $L/2$ denotes the central spin. In this plot with $[G^m_{L/2}(\tau)]^{-1/2x_m}$ on linear scale *versus* τ on a logarithmic scale the infinite system size limit is expected to lay on a straight line as indicated. (b) Same as (a) for the surface spin-spin autocorrelation function $G^m_1(\tau) = [\langle\sigma^x_1(\tau)\sigma^x_1\rangle]_{av}$ in imaginary time.



(a)



(b)

Fig. 4. (a) Bulk energy-energy autocorrelation function $G^e_{L/2}(\tau) = [\langle\sigma^z_{L/2}(\tau)\sigma^z_{L/2}\rangle]_{av}$ in imaginary time for various system sizes (and $\lambda = 4$) in a log-log plot. The straight line has slope -2.4, which yields our estimate for the exponent η_e. (b) Same as (a) for the surface energy-energy autocorrelation function $G^e_1(\tau) = [\langle\sigma^z_1(\tau)\sigma^z_1\rangle]_{av}$ in imaginary time. The straight line has slope -3.4, which yields our estimate for the exponent η^s_e.

the asymptotic decay is also consistent with a power law decay, as can be seen in Figure 4, both in the bulk and at the surface of the system. The corresponding exponents

$$\eta_e(\mathrm{RS}) \approx 2.4 \qquad \eta^s_e(\mathrm{RS}) \approx 3.4 \tag{4.25}$$

are, however, different from those of the random chain.

5 Off-critical properties

In random quantum systems there are Griffiths–McCoy singularities on the paramagnetic side of the critical point, which result in a power law decay of the autocorrelation function $G(\tau) \sim \tau^{-1/z(\delta)}$, where the dynamical exponent $z(\delta)$ is a continuous function of the control parameter δ. In the random TIM there is also a Griffiths-McCoy phase on the ferromagnetic side of the critical point and the values of the dynamical exponent in the two regions are related *via* duality, see paper I.

The long time behavior of the average autocorrelation function $G(\tau)$ is determined by the Laplace transform of the gap (ε_1) distribution function

$$G(\tau) \sim \int_0^\infty d\varepsilon\, P(\varepsilon) \exp(-\tau\varepsilon). \tag{5.1}$$

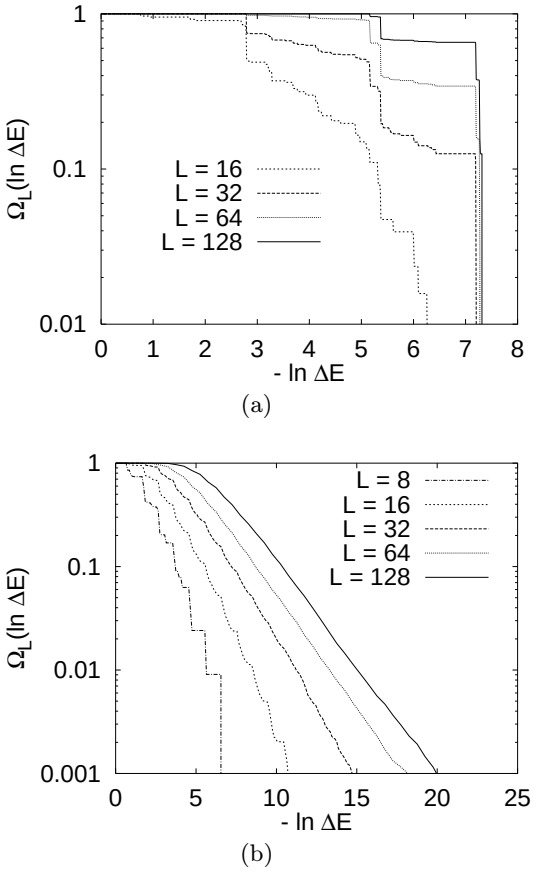


Fig. 5. (a) The integrated probability density $\Omega(\ln \Delta E)$ for the RS sequence (exact average) with $\lambda = 4$ slightly above the critical point ($h = 1.5$). The distribution is chopped off at $\ln \Delta E^{-1}_{min}(h = 1.5) \approx 7.3$. (b) The same as in (a) for the *random* chain with binary disorder, $\lambda = 4$. Note the larger x-range as compared with the aperiodic sequence and the absence of a cut-off for large enough system size. The asymptotic form of the distribution is $\Omega(x) \sim \exp(x/z(\delta))$, where $x = \ln \Delta E$ and $1/z(h_0 = 1.5) = 0.40$ (see paper I for details).

Thus the scaling properties of the low energy excitation are also connected to the above defined dynamical exponent $z(\delta)$:

$$\varepsilon(L, \delta) \propto L^{-z(\delta)} \tag{5.2}$$

in a finite system of length L.

For relevantly aperiodic chains the same type of scenario, *i.e.* the existence of Griffiths-McCoy singularities, have been speculated [21]. In the following we are going to clarify this issue and study the distribution function of the energy gap ε_1 in the disordered phase ($\delta > 0$) of the RS-chain. In Figure 5 we compare the integrated gap distribution functions for the random and RS-chains. While the data for the random chain follow the scaling prediction in (5.2) with $\delta = 0.5 \ln h \approx 0.20$, $z \approx 2.5$, the probability distribution of the RS-chain is chopped off: there is a L–independent cut-off at $\Delta E_{min}(\delta)$. Consequently there is a relevant time scale in the problem $t_r \sim \Delta E^{-1}_{min}(\delta)$ and the

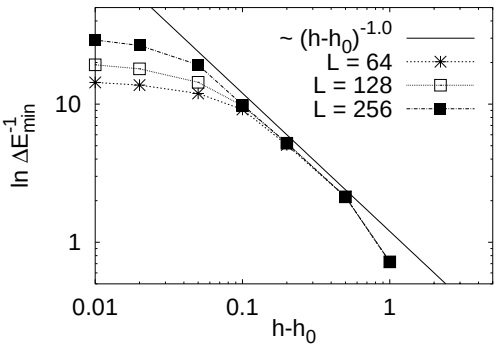


Fig. 6. The off-critical minimum energy gap of the RS-sequence $\ln \Delta E^{-1}_{min}(h)$ *versus* the distance from the critical point ($h - h_0 \sim \delta$ for $h \to h_0$) in the RS sequence for different system sizes. The straight line has slope -1, as predicted by (5.5).

autocorrelation function has an exponential decay. The susceptibility and other physical quantities are analytic in the whole disordered phase, thus there is *no Griffiths-McCoy region* in the RS-chain and we expect a similar behavior for any other aperiodic quantum spin chain.

We can estimate the minimum energy gap $\Delta E_{min}(\delta)$ as follows: We start with the formula for the excitation energy in (3.8) and consider a realization with surface order $m_s = \mathcal{O}(1)$ and $\overline{m}_s = \mathcal{O}(1)$, which is generally connected to the presence of a very small energy gap. Thus in the paramagnetic phase $0 < \delta \ll 1$ we have

$$\Delta E_{min}(\delta) \sim \varepsilon_1 \sim \prod_{i=1}^{L-1} \frac{h_i}{J_i} \sim \exp(A\delta L - BL^{\omega}), \tag{5.3}$$

where $A > 0$, $B > 0$. The first term in the exponential describes the average trend with δ, whereas the second represents the largest possible fluctuation in the couplings among all aperiodic sequences of length L. It is important to note that for a random distribution this second term could be proportional to L in rare events, implying that in the random case there is no minimum energy gap. For the aperiodic chains, however, due to the number of different realizations that increases only linearly with L the fluctuating energy (1.9) grows slower than L. Consequently from equation (5.3) one can derive a length scale

$$l_{ap} \sim \delta^{-1/(1-\omega)} \tag{5.4}$$

that characterizes the most singular sample and the corresponding minimum energy gap is then given by

$$\Delta E_{min}(\delta) \sim \exp\left(-const.\ \delta^{-\omega/(1-\omega)}\right). \tag{5.5}$$

This relation is indeed satisfied for the RS-chain, since according to our numerical results in this case $\ln \Delta E_{min}(\delta) \sim 1/\delta$, as can be seen in Figure 6.

6 Random walks in random and aperiodic environments

As it was shown in paper I and being utilized in Section 4.2 in this paper there is a close relation between the random quantum Ising spin chains and the one-dimensional random walk. Especially the scaling properties of the surface magnetization and that of the low energy excitations of the random TIM can be obtained from the surviving properties of a one-dimensional random walk. Here we go further and emphasize a relation between the random TIM and the 1d random walk in a random environment.

To be specific, we characterize the one-dimensional random walk with the nearest neighbor hopping by the transition probabilities $w_{i,j} = w(i \to j)$ for a random walker to jump from site i to site j with

$$w_{i,j} = \begin{cases} w_{i,i\pm1} & \text{for} \quad |i-j| = 1 \\ 0 & \text{for} \quad |i-j| > 1. \end{cases} \tag{6.1}$$

Here we are particularly interested in the general case, in which the transition probabilities are not necessarily symmetric, *i.e.*

$$w_{i,i+1} \neq w_{i+1,i}. \tag{6.2}$$

Moreover, the random walker is confined to a finite number of sites $i = 1, ..., L$. At the two ends of this interval, *i.e.* at $i = 0$ and $i = L+1$, we put *absorbing walls*, which is simply modeled by setting $w_{0,1} = w_{L+1,L} = 0$ (*i.e.* the walker cannot jump back into the system once landed on 0 or $L+1$). The time evolution of the probability distribution of the walk $P_{i,j}(t)$, which is the probability for the walker to be at time t on site j once started at time 0 on site i, is fully determined by the Master-equation

$$\frac{d}{dt}\,\underline{P}(t) = \underline{\underline{M}} \cdot \underline{P}(t). \tag{6.3}$$

Here

$$\underline{P}(t) = (P_{i,0}(t), P_{i,1}(t), \ldots, P_{i,L}(t), P_{i,L+1}(t))^T \tag{6.4}$$

and the transition matrix is $(\underline{\underline{M}})_{i,j} = w_{i,j}$ for $i \neq j$ and $(\underline{\underline{M}})_{i,i} = -\sum_j w_{i,j}$ while the initial condition is $P_{i,j}(0) = \delta_{i,j}$. All physical properties of the model can be expressed in terms of (left and right) eigenvectors and eigenvalues of $\underline{\underline{M}}$, very much in the same way as the physics of the TIM is contained in the eigenvectors and eigenvalues of the tridiagonal matrix (3.2).

Here we consider first one quantity that gained considerable interest recently in related models for anomalous diffusion [35,36]: The *persistence probability* $P_{pr}(L,t)$, which is the probability that a walker starting at site $i = 1$ does not cross its starting point until time t. Due to the absorbing sites at $i = 0$ and $i = L+1$ its long time limit $p_{pr}(L) = \lim_{t\to\infty} P_{pr}(L,t)$ is simply given by [37]

$$p_{pr}(L) = \lim_{t\to\infty} P_{1,L+1}(t) = \left(1 + \sum_{i=1}^{L}\prod_{j=1}^{i}\frac{w_{j,j-1}}{w_{j,j+1}}\right)^{-1}. \tag{6.5}$$

Thus, as is shown in [37], there is a one-to-one relation between the persistence probability (6.5) and the surface magnetization $m_s(L)$ of the TIM (3.6) with the following correspondences

$$\begin{aligned} w_{i,i+1} &\longrightarrow J_i^2 \\ w_{i,i-1} &\longrightarrow h_i^2 \\ p_{pr}(L) &\longrightarrow m_s^2(L). \end{aligned} \tag{6.6}$$

Consequently similar relations hold for the average quantities, when the transition probabilities (or equivalently the fields and the couplings) follow the same random or aperiodic modulation.

In the random case the critical point of the TIM corresponds to the Sinai walk [1], and from equations (3.6, 4.6, 6.5) we have

$$[p_{pr}^{(rand)}(L)]_{av} \propto L^{-1/2}. \tag{6.7}$$

For relevantly aperiodic environments which ar characterized by a wandering exponent $\omega > 0$ we get from (4.10)

$$[p_{pr}^{(aperiodic)}(L)]_{av} \propto L^{-(1-\omega)}. \tag{6.8}$$

In the non-critical situation there is an average drift of the walk, which can be defined through equation (1.2) as

$$\delta_{RW} = \frac{[\ln w_{\rightarrow}]_{av} - [\ln w_{\leftarrow}]_{av}}{var[\ln w_{\rightarrow}] + var[\ln w_{\leftarrow}]}, \tag{6.9}$$

where $w_{\rightarrow}$ ($w_{\leftarrow}$) stands for transition probabilities to the right (left), *i.e.* $w_{i,i+1}$ ($w_{i,i-1}$). For $\delta_{RW} \neq 0$ the average correlations defined on persistent walks are characterized by a correlation length

$$\xi \sim |\delta_{RW}|^{-\nu}, \tag{6.10}$$

with the exponents for the *average* given in equations (1.4, 4.16) for the random and aperiodic environments, respectively.

The dynamical properties of the random walk are dominated by the largest, non-vanishing eigenvalue λ_m of the transition matrix $\underline{\underline{M}}$. It can be shown [37] that for λ_m a formula similar to equation (3.8) holds, where h_i and J_i have to be replaced by their random walk counterparts given in (6.6) and for the term $m_s\overline{m_s}$ stands the persistence probability $p_{pr}(L)$. This implies straightforwardly, using equation (4.2) that in a Sinai walk in the random and relevantly aperiodic case the characteristic time scales like

$$t_{char}^{(rand)} \sim \left(\lambda_m^{(rand)}\right)^{-1} \sim \exp(const.\ L^{\omega}) \tag{6.11}$$

with $\omega = 1/2$ in the random case. As a further consequence, autocorrelations or return probabilities decay logarithmically,

$$P_0(t) = \frac{1}{L}\sum_{i=1}^{L} P_{i,i}(t) \propto \frac{1}{\ln^{\eta}(t)} \tag{6.12}$$

with $\eta = 2$ in the random case and $\eta = 1/\omega$ in the relevantly aperiodic case. This follows from the scaling result $P_0(t)^2 \sim \left[X^2(t)\right]_{av}^{-1}$, where $\left[X^2(t)\right]_{av}$ is the average mean-square displacement, as already mentioned in the Introduction.

Finally the Griffiths-McCoy phase of the *random* TIM is equivalent to the anomalous diffusion region of the random walk with $\delta_{RW} \neq 0$, in which case autocorrelations decay anomalously slow with an exponent $\gamma(\delta_{RW}) \leq 1$

$$P_0(t) \sim t^{-\gamma(\delta_{RW})} \tag{6.13}$$

that depends continuously on the drift parameter $\delta_{\rm RW}$ and corresponds to the inverse dynamical exponent $z(\delta)$ of the Griffiths-McCoy phase of the random TIM. In the two limiting cases we have $z(\delta) = 1/2\delta$ as $|\delta| \to 0$ [7] and $z(\delta) = 1$ as $|\delta| \to \infty$. In the latter case we approach the ballistic situation, when all the steps of the walk are made in the same direction, thus the average displacement has a linear time dependence. We can thus extend our dictionary (6.6) by

$$\begin{aligned} \lambda_m &\longrightarrow \varepsilon_1^2 \\ \gamma(\delta_{RW}) &\longrightarrow 1/(2z(\delta)). \end{aligned} \tag{6.14}$$

For the *aperiodic* case with drift we remember the results of Section 5, in particular equation (5.5) and conclude that a random walk in a relevantly aperiodic environment for $\delta_{RW} \neq 0$ does *not* exhibit a region of anomalous diffusion.

7 Summary

To summarize we have studied the effect of random and aperiodic environments on cooperative processes in one space dimension. We have shown that at the critical point, both for the transverse-field Ising model and for the diffusion process, the two types of inhomogeneities have quite similar consequences, which is based on the same type of distribution of the low energy excitations (large time scales). We have obtained – presumably exact – scaling relations, which connect the values of the surface magnetization exponent and that of the correlation length exponent with the known characteristics of the (random and aperiodic) environments. Besides the similarities between the critical properties of random and aperiodic models we have also observed several quantitative differences. For example some critical exponents are turned out to be environment dependent and – most noticeably – the Griffiths phase is absent for aperiodic models.

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Appendix: Average critical surface magnetization for the Rudin-Shapiro chain

We consider the RS-chain in (1.10) generated from a letter A and calculate the surface magnetization (3.6) for chains of length $L = 2^{2l+1}$, l =1,2, ... The chain starts at the 1st, 2nd, ..., Nth position of the original RS-chain, such that $N = L\ 2^{2k}$, k =1,2, ..., and average over these N realizations. The average critical surface magnetization in the limit $\lambda \to \infty$ is given by:

$$\begin{aligned} m_s(l,k) = \frac{1}{N}\Bigg\{ & N_1(l,k) + \frac{N_2(l,k)}{\sqrt{2}} + \frac{N_u(l,k)}{\sqrt{2^k+1}} \\ & + H(k)[S_1(l) + S_2(l)] \Bigg\} \end{aligned} \tag{A.1}$$

where

$$S_1(l) = \sum_{m=1}^{l-1} 2(l-m) \sum_{n=1}^{2^{m-1}} \frac{1}{\sqrt{2^{m-1}+1+n}} \tag{A.2}$$

$$S_2(l) = \sum_{n=1}^{2^{l-1}-1} \frac{1}{\sqrt{2^{k-1}+1+n}} \tag{A.3}$$

$$\begin{aligned} N_1(l,k) = {} & 18 \cdot 4^{k-2} + 9 \cdot 2^{k-2} \\ & + 5\,(2^{l-1}-1)(4^{k-1}+2^{k-1}) \end{aligned} \tag{A.4}$$

$$\begin{aligned} N_2(l,k) = {} & 4^{k-2}\Big(10 \cdot 2^{l-1} + 4(l-1)\Big) \\ & + \Big(5 \cdot 2^{l-1} + 2(l-1) - 2\Big) \cdot 2^{k-2} \end{aligned} \tag{A.5}$$

$$N_u(l,k) = 2^{2k-1} + (4^{k-1} - 2^{k-1})(2^{l-1}-1) \tag{A.6}$$

$$H(k) = 2 \cdot 4^{k-2} + 2^{k-2}. \tag{A.7}$$

The asymptotic behavior of $\lim_{k\to\infty} m_s(l,k)$ is given in equation (4.7).

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